

Noether's problem for the groups with a cyclic subgroup of index 4

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Abstract. Let G be a finite group and k be a field. Let G act on the rational function field $k(x_g : g \in G)$ by k -automorphisms defined by $g \cdot x_h = x_{gh}$ for any $g, h \in G$. Noether's problem asks whether the fixed field $k(G) = k(x_g : g \in G)^G$ is rational (i.e. purely transcendental) over k . Theorem 1. If G is a group of order 2^n ($n \geq 4$) and of exponent 2^e such that (i) $e \geq n - 2$ and (ii) $\zeta_{2^{e-1}} \in k$, then $k(G)$ is k -rational. Theorem 2. Let G be a group of order $4n$ where n is any positive integer (it is unnecessary to assume that n is a power of 2). Assume that (i) $\text{char } k \neq 2$, $\zeta_n \in k$, and (ii) G contains an element of order n . Then $k(G)$ is rational over k , except for the case $n = 2m$ and $G \simeq C_m \rtimes C_8$ where m is an odd integer and the center of G is of even order (note that C_m is normal in $C_m \rtimes C_8$); for the exceptional case, $k(G)$ is rational over k if and only if at least one of $-1, 2, -2$ belongs to $(k^\times)^2$.

2010 Mathematics Subject Classification. Primary 13A50, 14E08, 14M20, 12F12.

Keywords: Noether's problem, rationality problem, the inverse Galois problem.

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¹ Both authors were partially supported by National Center for Theoretic Sciences (Taipei Office).

² Partially supported by project No. RD-05-156/25.02.2011 of Shumen University.

³ The work of this paper was finished when the third-named author visited National Taiwan University under the support by National Center for Theoretic Sciences (Taipei Office).

§1. Introduction

Let k be any field and G be a finite group. Let G act on the rational function field $k(x_g : g \in G)$ by k -automorphisms such that $g \cdot x_h = x_{gh}$ for any $g, h \in G$. Denote by $k(G)$ the fixed field $k(x_g : g \in G)^G$. Noether's problem asks whether $k(G)$ is rational (=purely transcendental) over k . It is related to the inverse Galois problem, to the existence of generic G -Galois extensions over k , and to the existence of versal G -torsors over k -rational field extensions [Sw; Sa1; GMS, 33.1, p.86]. Noether's problem for abelian groups was studied extensively by Swan, Voskresenskii, Endo, Miyata and Lenstra, etc. The reader is referred to Swan's paper for a survey of this problem [Sw].

On the other hand, just a handful of results about Noether's problem are obtained when the groups are not abelian. It is the case even when the group G is a p -group. The reader is referred to [CK; Ka3; HuK; Ka6] for previous results of Noether's problem for p -groups. In the following we will list only those results pertaining to the 2-groups.

Theorem 1.1 (Chu, Hu and Kang [CHK; Ka2]) *Let k be any field. Suppose that G is a non-abelian group of order 8 or 16. Then $k(G)$ is rational over k , except when $\text{char } k \neq 2$ and $G = Q_{16}$, the generalized quaternion group of order 16. When $\text{char } k \neq 2$ and $G = Q_{16}$, then $k(G)$ is also rational over k provided that $k(\zeta_8)$ is a cyclic extension over k where ζ_8 is a primitive 8-th root of unity.*

Theorem 1.2 (Serre [GMS, Theorem 34.7]) *If $G = Q_{16}$, then $\mathbb{Q}(G)$ is not stably rational over \mathbb{Q} ; in particular, it is not rational over \mathbb{Q} .*

We don't know the answer whether $k(G)$ is rational over k or not, if $G = Q_{16}$ and k is any field other than \mathbb{Q} such that $k(\zeta_8)$ is not a cyclic extension of k . The reader is referred to [CHKP; CHKK] for groups of order 32 and 64.

Among the known results of Noether's problem for non-abelian p -groups, except the situations in Theorem 1.1, assumptions on the existence of "enough" roots of unity always arose (see, for example, the following Theorem 1.3). In fact, even when G is a non-abelian p -group of order p^3 where p is an odd prime number, it is not known how to find a necessary and sufficient condition such as $\mathbb{Q}(G)$ is rational over \mathbb{Q} (see [Ka5]). Similarly, without assuming the existence of roots of unity, we don't have a good criterion to guarantee $\mathbb{Q}(G)$ is rational where G is a non-abelian group of order 32 (compare with the results in [CHKP]). Thus it will be desirable if we can weaken the assumptions on the existence of roots of unity.

Now we turn to metacyclic p -groups. For results of a general metacyclic p -group, see [Ka3]. The following theorem provides a sharper result for metacyclic p -groups with a large cyclic subgroup. We state the result for the 2-groups only.

Theorem 1.3 (Hu and Kang [HuK; Ka6]) *Let $n \geq 4$ and G be a non-abelian group of order 2^n . Assume that either (i) $\text{char } k = 2$, or (ii) $\text{char } k \neq 2$ and k contains a primitive 2^{n-2} -th root of unity. If G contains an element whose order $\geq 2^{n-2}$, then $k(G)$ is rational over k .*

The purpose of this paper is to generalize the above Theorem 1.3 in two directions. The first main result is the following theorem.

Theorem 1.4 *Let $n \geq 4$ and G be a group of order 2^n and of exponent 2^e where $e \geq n - 2$. Assume that either (i) $\text{char } k = 2$, or (ii) $\text{char } k \neq 2$ and k contains a primitive 2^{e-1} -th root of unity. Then $k(G)$ is rational over k .*

For the second generalization of Theorem 1.3, recall a theorem of Michailov [Mi]. Imitating the definition of the 2-groups modular groups, dihedral groups, quasi-dihedral groups, generalized quaternion groups (see [HuK, Theorem 1.9]), the groups $M_{8n}, D_{8n}, SD_{8n}, Q_{8n}$ are defined in [Mi, Section 1] where n is any positive integer and it is unnecessary to assume that n is a power of 2. Denote by H anyone of these groups. Note that H is a group of order $8n$ containing an element of order $4n$. The following theorem generalizes Theorem 1.3 in another direction.

Theorem 1.5 (Michailov [Mi, Theorem 1.2]) *Let H be a group of order $8n$ defined above, and G be a central extension of H defined by $1 \rightarrow C_2 \rightarrow G \rightarrow H \rightarrow 1$ where C_2 is the cyclic group of order two. If k is a field containing a primitive $4n$ -th root of unity. Then $k(G)$ is rational over k .*

The second main result of this paper is the following theorem which generalizes Theorem 1.3 and Theorem 1.5.

Theorem 1.6 *Let G be a group of order $4n$ where n is any positive integer (it is unnecessary to assume that n is a power of 2). Assume that (i) $\text{char } k \neq 2$ and k contains a primitive n -th root of unity, and (ii) G contains an element of order n . Then $k(G)$ is rational over k , except for the case $n = 2m$ and $G \simeq C_m \rtimes C_8$ where m is an odd integer and the center of G is of even order (note that C_m is normal in $C_m \rtimes C_8$); for the exceptional case, $k(G)$ is rational over k if and only if at least one of $-1, 2, -2$ belongs to $(k^\times)^2$.*

The main tool in proving the above Theorem 1.6 is Theorem 2.5 and Theorem 2.6 in Section 2 of this paper. Theorem 2.5 and Theorem 2.6 are consequences of Yamasaki's paper [Ya] and the paper by Hoshi, Kitayama and Yamasaki [HKY]. Without these two papers, the proof of Theorem 1.6 would be very involved.

The proof of Theorem 1.4 is more computational. It relies on the classification of 2-groups with a cyclic subgroup of index 4 [Ni]. Note that in order to prove Theorem 1.4 we may assume the following extra conditions on G and k without loss of generality

$$(1.1) \quad n \geq 5, \quad |G| = 2^n, \quad \exp(G) = 2^{n-2}, \quad G \text{ is non-abelian, } \text{char } k \neq 2 \text{ and } \zeta_{2^{n-3}} \in k.$$

For, it is not difficult to prove Theorem 1.4 when G is an abelian group by applying Lenstra's Theorem [Le]. Moreover, Kuniyoshi's Theorem asserts that, if $\text{char } k = p > 0$ and G is a p -group, then $k(G)$ is rational over k [Ku; KP, Corollary 1.2]. Thus we may assume that G is non-abelian and $\text{char } k \neq 2$. When G is a non-abelian group of order 2^n , the case of Theorem 1.4 when $n = 4$ is taken care by Theorem 1.1, and the case

when $\exp(G) = 2^{n-1}$ is taken care by Theorem 1.3. Thus only the situation of (1.1) remains.

The key idea to prove Theorem 1.4 is, by applying Theorem 2.2, to find a low-dimensional faithful G -subspace $W = \bigoplus_{1 \leq i \leq m} k \cdot y_i$ of the regular representation space $\bigoplus_{g \in G} k \cdot x(g)$ and to show that $k(y_i : 1 \leq i \leq m)^G$ is rational over k . The subspace W is obtained as an induced representation from some abelian subgroup of G . This method is reminiscent of some techniques exploited in [Ka6]. However, the proof of Theorem 1.4 is more subtle and requires elaboration. For examples, in [Ka6], the following two theorems were used to solve the rationality problem for many groups G_i in Theorem 2.1.

Theorem 1.7 ([Ka3]) *Let k be a field and G be a metacyclic p -group. Assume that (i) $\text{char } k = p > 0$, or (ii) $\text{char } k \neq p$ and $\zeta_e \in k$ where $e = \exp(G)$. Then $k(G)$ is rational over k .*

Theorem 1.8 ([Ka4, Theorem 1.4]) *Let k be a field and G be a finite group. Assume that (i) G contains an abelian normal subgroup H so that G/H is cyclic of order n , (ii) $\mathbb{Z}[\zeta_n]$ is a unique factorization domain, and (iii) $\zeta_e \in k$ where e is the exponent of G . If $G \rightarrow GL(V)$ is any finite-dimensional linear representation of G over k , then $k(V)^G$ is rational over k .*

Because we assume $\zeta_{2^{n-3}} \in k$ (instead of $\zeta_{2^{n-2}} \in k$) in (1.1), the above two theorems are not directly applicable in the present situation. This is the reason why we should find judiciously a faithful subspace W . Fortunately we can find these subspaces W in an almost unified way. In fact, the proof for the group G_8 in Theorem 2.1 is a typical case; the proof for other groups is either similar to that of G_8 or has appeared in [Ka6].

We organize this paper as follows. In Section 2 we recall Ninomiya's classification of non-abelian groups G with $|G| = 2^n$ and $\exp(G) = 2^{n-2}$ (where $n \geq 4$). We also recall some preliminaries which will be used in the proof of Theorem 1.4 and Theorem 1.6. The proof of Theorem 1.4 is given in Section 3. In Section 4 the proof of Theorem 1.6 is given. Note that Theorem 4.4 is of interest itself.

Standing Notations. Throughout this article, $k(x_1, \dots, x_n)$ or $k(x, y)$ will be rational function fields over k . ζ_n denotes a primitive n -th root of unity in some algebraic extension of the field k . Whenever we write $\zeta_n \in k$, it is understood that either $\text{char } k = 0$ or $\text{char } k = p > 0$ with $\gcd\{p, n\} = 1$.

A field extension L of k is called rational over k (or k -rational, for short) if $L \simeq k(x_1, \dots, x_n)$ over k for some integer n . L is stably rational over k if $L(y_1, \dots, y_m)$ is rational over k for some y_1, \dots, y_m which are algebraically independent over L . Recall that, if G is a finite group, $k(G)$ denotes $k(x_g : g \in G)^G$ where $h \cdot x_g = x_{hg}$ for $h, g \in G$.

The exponent of a finite group G , denoted by $\exp(G)$, is $\text{lcm}\{\text{ord}(g) : g \in G\}$ where $\text{ord}(g)$ is the order of g . The cyclic group of order n is denoted by C_n .

If G is a group acting on a rational function field $k(x_1, \dots, x_n)$ by k -automorphisms, the actions of G are called purely monomial actions if, for any $\sigma \in G$, any $1 \leq j \leq n$,

$\sigma \cdot x_j = \prod_{1 \leq i \leq n} x_i^{a_{ij}}$ where $a_{ij} \in \mathbb{Z}$; similarly, the actions of G are called monomial actions if, for any $\sigma \in G$, any $1 \leq j \leq n$, $\sigma \cdot x_j = \lambda_j(\sigma) \cdot \prod_{1 \leq i \leq n} x_i^{a_{ij}}$ where $a_{ij} \in \mathbb{Z}$ and $\lambda_j(\sigma) \in k \setminus \{0\}$. When G acts on $k(x_1, \dots, x_n)$ by monomial actions and $\alpha \in k \setminus \{0\}$, we say that G acts on $k(x_1, \dots, x_n)$ by monomial actions with coefficients in $\langle \alpha \rangle$ if $\lambda_j(\sigma) \in \langle \alpha \rangle$ for all $\sigma \in G$, for all $1 \leq j \leq n$ (here $\langle \alpha \rangle$ denotes the multiplicative subgroup in $k \setminus \{0\}$ generated by α). All the groups in this article are finite groups.

§2. Preliminaries

Theorem 2.1 (Ninomiya [Ni, Theorem 2]) *Let $n \geq 4$. The finite non-abelian groups of order 2^n which have a cyclic subgroup of index 4, but haven't a cyclic subgroup of index 2 are of the following types:*

(I) $n \geq 4$

$$\begin{aligned} G_1 &= \langle \sigma, \tau : \sigma^{2^{n-2}} = \tau^4 = 1, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-3}} \rangle, \\ G_2 &= \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \lambda^2 = 1, \sigma^{2^{n-3}} = \tau^2, \tau^{-1}\sigma\tau = \sigma^{-1}, \sigma\lambda = \lambda\sigma, \tau\lambda = \lambda\tau \rangle, \\ G_3 &= \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1}, \sigma\lambda = \lambda\sigma, \tau\lambda = \lambda\tau \rangle, \\ G_4 &= \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \sigma\tau = \tau\sigma, \sigma\lambda = \lambda\sigma, \lambda^{-1}\tau\lambda = \sigma^{2^{n-3}}\tau \rangle, \\ G_5 &= \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \sigma\tau = \tau\sigma, \lambda^{-1}\sigma\lambda = \sigma\tau, \tau\lambda = \lambda\tau \rangle. \end{aligned}$$

(II) $n \geq 5$

$$\begin{aligned} G_6 &= \langle \sigma, \tau : \sigma^{2^{n-2}} = \tau^4 = 1, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle, \\ G_7 &= \langle \sigma, \tau : \sigma^{2^{n-2}} = \tau^4 = 1, \tau^{-1}\sigma\tau = \sigma^{-1+2^{n-3}} \rangle, \\ G_8 &= \langle \sigma, \tau : \sigma^{2^{n-2}} = 1, \sigma^{2^{n-3}} = \tau^4, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle, \\ G_9 &= \langle \sigma, \tau : \sigma^{2^{n-2}} = \tau^4 = 1, \sigma^{-1}\tau\sigma = \tau^{-1} \rangle, \\ G_{10} &= \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-3}}, \sigma\lambda = \lambda\sigma, \tau\lambda = \lambda\tau \rangle, \\ G_{11} &= \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1+2^{n-3}}, \sigma\lambda = \lambda\sigma, \tau\lambda = \lambda\tau \rangle, \\ G_{12} &= \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \sigma\tau = \tau\sigma, \lambda^{-1}\sigma\lambda = \sigma^{-1}, \lambda^{-1}\tau\lambda = \sigma^{2^{n-3}}\tau \rangle, \\ G_{13} &= \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \sigma\tau = \tau\sigma, \lambda^{-1}\sigma\lambda = \sigma^{-1}\tau, \tau\lambda = \lambda\tau \rangle, \\ G_{14} &= \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = 1, \sigma^{2^{n-3}} = \lambda^2, \sigma\tau = \tau\sigma, \lambda^{-1}\sigma\lambda = \sigma^{-1}\tau, \tau\lambda = \lambda\tau \rangle, \\ G_{15} &= \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-3}}, \lambda^{-1}\sigma\lambda = \sigma^{-1+2^{n-3}}, \tau\lambda = \lambda\tau \rangle, \\ G_{16} &= \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-3}}, \lambda^{-1}\sigma\lambda = \sigma^{-1+2^{n-3}}, \\ &\quad \lambda^{-1}\tau\lambda = \sigma^{2^{n-3}}\tau \rangle, \\ G_{17} &= \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-3}}, \lambda^{-1}\sigma\lambda = \sigma\tau, \tau\lambda = \lambda\tau \rangle, \\ G_{18} &= \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = 1, \lambda^2 = \tau, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-3}}, \lambda^{-1}\sigma\lambda = \sigma^{-1}\tau \rangle. \end{aligned}$$

(III) $n \geq 6$

$$\begin{aligned}
G_{19} &= \langle \sigma, \tau : \sigma^{2^{n-2}} = \tau^4 = 1, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-4}} \rangle, \\
G_{20} &= \langle \sigma, \tau : \sigma^{2^{n-2}} = \tau^4 = 1, \tau^{-1}\sigma\tau = \sigma^{-1+2^{n-4}} \rangle, \\
G_{21} &= \langle \sigma, \tau : \sigma^{2^{n-2}} = 1, \sigma^{2^{n-3}} = \tau^4, \sigma^{-1}\tau\sigma = \tau^{-1} \rangle, \\
G_{22} &= \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \sigma\tau = \tau\sigma, \lambda^{-1}\sigma\lambda = \sigma^{1+2^{n-4}}\tau, \lambda^{-1}\tau\lambda = \sigma^{2^{n-3}}\tau \rangle, \\
G_{23} &= \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \sigma\tau = \tau\sigma, \lambda^{-1}\sigma\lambda = \sigma^{-1+2^{n-4}}\tau, \lambda^{-1}\tau\lambda = \sigma^{2^{n-3}}\tau \rangle, \\
G_{24} &= \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-3}}, \lambda^{-1}\sigma\lambda = \sigma^{-1+2^{n-4}}, \tau\lambda = \lambda\tau \rangle, \\
G_{25} &= \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = 1, \sigma^{2^{n-3}} = \lambda^2, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-3}}, \lambda^{-1}\sigma\lambda = \sigma^{-1+2^{n-4}}, \\
&\quad \tau\lambda = \lambda\tau \rangle,
\end{aligned}$$

(IV) $n = 5$

$$G_{26} = \langle \sigma, \tau, \lambda : \sigma^8 = \tau^2 = 1, \sigma^4 = \lambda^2, \tau^{-1}\sigma\tau = \sigma^5, \lambda^{-1}\sigma\lambda = \sigma\tau, \tau\lambda = \lambda\tau \rangle.$$

Theorem 2.2 ([HK2, Theorem 1]) *Let G be a finite group acting on $L(x_1, \dots, x_n)$, the rational function field of n variables over a field L . Suppose that*

- (i) *for any $\sigma \in G$, $\sigma(L) \subset L$;*
- (ii) *the restriction of the action of G to L is faithful;*
- (iii) *for any $\sigma \in G$,*

$$\begin{pmatrix} \sigma(x_1) \\ \sigma(x_2) \\ \vdots \\ \sigma(x_n) \end{pmatrix} = A(\sigma) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + B(\sigma)$$

where $A(\sigma) \in GL_n(L)$ and $B(\sigma)$ is an $n \times 1$ matrix over L .

Then there exist elements $z_1, \dots, z_n \in L(x_1, \dots, x_n)$ such that $L(x_1, \dots, x_n) = L(z_1, \dots, z_n)$ and $\sigma(z_i) = z_i$ for any $\sigma \in G$, any $1 \leq i \leq n$.

Theorem 2.3 ([AHK, Theorem 3.1]) *Let L be any field, $L(x)$ the rational function field of one variable over L , and G a finite group acting on $L(x)$. Suppose that, for any $\sigma \in G$, $\sigma(L) \subset L$ and $\sigma(x) = a_\sigma \cdot x + b_\sigma$ where $a_\sigma, b_\sigma \in L$ and $a_\sigma \neq 0$. Then $L(x)^G = L^G(f)$ for some polynomial $f \in L[x]$. In fact, if $m = \min\{\deg g(x) : g(x) \in L[x]^G \setminus L\}$, any polynomial $f \in L[x]^G$ with $\deg f = m$ satisfies the property $L(x)^G = L^G(f)$.*

Theorem 2.4 (Hoshi, Kitayama and Yamasaki [HKY, 5.4]) *Let k be a field with $\text{char } k \neq 2$, $\varepsilon \in \{1, -1\}$ and $a, b \in k \setminus \{0\}$. Let $G = \langle \sigma, \tau \rangle$ act on $k(x, y, z)$ by k -automorphisms defined by*

$$\begin{aligned}
\sigma : x &\mapsto a/x, \quad y \mapsto a/y, \quad z \mapsto \varepsilon z, \\
\tau : x &\mapsto y \mapsto x, \quad z \mapsto b/z.
\end{aligned}$$

Then $k(x, y, z)^G$ is rational over k .

Theorem 2.5 *Let k be a field with $\text{char } k \neq 2$, α be a non-zero element in k . Let G be a finite group acting on $k(x_1, x_2, x_3)$ by monomial k -automorphisms with coefficients in $\langle \alpha \rangle$. Assume that $\sqrt{-1}$ belongs to k . Then $k(x_1, x_2, x_3)^G$ is rational over k .*

Proof. Recall the definition of monomial actions with coefficients in $\langle \alpha \rangle$ in the last paragraph of Section 1.

For a monomial action of G on $k(x_1, x_2, x_3)$, define the group homomorphism $\rho_x : G \rightarrow GL_3(\mathbb{Z})$ by $\rho_x(\sigma) = (m_{ij})_{1 \leq i, j \leq 3}$ if $\sigma \cdot x_j$ is given by $\sigma \cdot x_j = \lambda_j(\sigma) \cdot \prod_{1 \leq i \leq 3} x_i^{m_{ij}}$ where $m_{ij} \in \mathbb{Z}$ and $\lambda_j(\sigma) \in k \setminus \{0\}$ (see [KPr, Definition 2.7]).

By [KPr, Lemma 2.8], we can find a normal subgroup H such that (i) $k(x_1, x_2, x_3)^H = k(z_1, z_2, z_3)$ with each one of z_1, z_2, z_3 in the form $x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3}$ (where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}$), (ii) G/H acts on $k(z_1, z_2, z_3)$ by monomial k -automorphisms, and (iii) $\rho_z : G/H \rightarrow GL_3(\mathbb{Z})$ is injective.

Clearly G/H acts on $k(z_1, z_2, z_3)$ by monomial actions with coefficients in $\langle \alpha \rangle$ also. Thus we may apply the results of [Ya; HKY] to $k(z_1, z_2, z_3)^{G/H}$.

It is known that the fixed field of any 3-dimensional monomial action on $k(z_1, z_2, z_3)$ is rational except possibly for nine cases [HKY, Theorem 1.6].

The first eight “exceptional” cases are studied in [Ya]. Because of our assumptions that $\sqrt{-1}$ and the monomial actions have coefficients in $\langle \alpha \rangle$, the rationality criteria found by Yamasaki are satisfied. For example, in [Ya, Lemma 2], $R(a, b, c)$ is affirmative if $a, b, c \in \langle \alpha \rangle$; for, if both a and b are in $\langle \alpha \rangle \setminus \langle \alpha^2 \rangle$, then $ab \in \langle \alpha^2 \rangle$.

The last “exceptional” case is studied in [HKY, Theorem 1.7], which is rescued by the assumption $\sqrt{-1} \in k$. \square

Theorem 2.6 *Let k be a field with $\text{char } k \neq 2$, ζ_m be a primitive m -th root of unity belonging to k . Let G be a finite group acting on $k(x_1, x_2, x_3)$ by monomial k -automorphisms with coefficients in $\langle \zeta_m \rangle$. Then $k(x_1, x_2, x_3)^G$ is rational over k except for the following situation : there is a normal subgroup H of G such that $G/H = \langle \tau \rangle$ is cyclic of order 4, $k(x_1, x_2, x_3)^H = k(z_1, z_2, z_3)$, each one of z_1, z_2, z_3 is of the form $x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3}$ (where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}$), $-1 \in \langle \zeta_m \rangle$ and $\tau : z_1 \mapsto z_2 \mapsto z_3 \mapsto -1/(z_1 z_2 z_3)$.*

For the exceptional case, $k(x_1, x_2, x_3)^G$ is rational over k if and only if at least one of $-1, 2, -2$ belongs to $(k^\times)^2$.

Proof. The proof is very similar to that of Theorem 2.5. By [KPr, Lemma 2.8], find the normal subgroup H of G such that (i) $k(x_1, x_2, x_3)^H = k(z_1, z_2, z_3)$ with each one of z_1, z_2, z_3 in the form $x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3}$ (where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}$), (ii) G/H acts on $k(z_1, z_2, z_3)$ by monomial k -automorphisms, and (iii) $\rho_z : G/H \rightarrow GL_3(\mathbb{Z})$ is injective. Thus we may apply the results of [Ya; HKY] to $k(z_1, z_2, z_3)^{G/H}$.

Clearly G/H acts on $k(z_1, z_2, z_3)$ by monomial actions with coefficients in $\langle \zeta_m \rangle$ also.

By [HKY, Theorem 1.6] it remains to check the “nine cases” as in the proof of Theorem 2.5. Note that, if m is an odd integer, then $-\zeta_m$ is a primitive $2m$ -th root of unity in k and $\zeta_m \in (k^\times)^2$.

We may also assume $\sqrt{-1} \notin k$ and thus 4 doesn't divide m ; otherwise, apply Theorem 2.5 and the proof is finished. From now on, assume either m is odd or $m = 2s$ for some odd integer s .

Apply the results of [Ya; HKY]. All the cases are rational except the group $G_{4,2,2}$ in Section 4 of [Ya]. This group, G/H , is a cyclic group of order 4 and the group action is given by $\tau : z_1 \mapsto z_2 \mapsto z_3 \mapsto c/(z_1 z_2 z_3)$ for some $c \in \langle \zeta_m \rangle$.

If m is odd and $c \in \langle \zeta_m \rangle$, we may write $c = d^4$ for some $d \in \langle \zeta_m \rangle$ (because $\gcd\{m, 4\} = 1$). Define $w_j = z_j/d$ for $1 \leq j \leq 3$. We find that $\tau : w_1 \mapsto w_2 \mapsto w_3 \mapsto 1/(w_1 w_2 w_3)$. We get a purely monomial action and $k(x_1, x_2, x_3)^G = k(w_1, w_2, w_3)^{<\tau>}$ is rational by [HK1].

Now consider the case where $m = 2s$ for some odd integer s .

Since $c \in \langle \zeta_{2s} \rangle \simeq C_2 \times C_s$, we may write $c = \varepsilon b$ where $\varepsilon \in \{1, -1\}$ and $b^s = 1$. As $b^s = 1$, we may write $b = a^4$ for some $a \in k^\times$. Define $w_j = z_j/a$ for $1 \leq j \leq 3$. We find that $\tau : w_1 \mapsto w_2 \mapsto w_3 \mapsto \varepsilon/(w_1 w_2 w_3)$.

If $\varepsilon = 1$, we get a purely monomial action and $k(w_1, w_2, w_3)^{<\tau>}$ is rational by [HK1]. Thus only the case $\varepsilon = -1$ remains. Note that $-1 \in \langle \zeta_m \rangle$ in the present situation.

Now we will solve the “exceptional” case whether $k(x_1, x_2, x_3)^G = k(w_1, w_2, w_3)^{<\tau>}$ is rational over k with $\tau : w_1 \mapsto w_2 \mapsto w_3 \mapsto -1/(w_1 w_2 w_3)$.

Assume that $k(w_1, w_2, w_3)^{<\tau>}$ is rational. The desired conclusion “at least one of $-1, 2, -2$ belongs to $(k^\times)^2$ ”, which is equivalent to “ $k(\zeta_8)$ is cyclic over k ”, is valid when k is a finite field. Hence we may assume that k is an infinite field.

Consider the fixed field $k(C_8)$ where $k(C_8) = k(y_i : 1 \leq i \leq 8)^{<\tau>}$ with $\tau : y_1 \mapsto y_2 \mapsto \dots \mapsto y_8 \mapsto y_1$. We will show that $k(C_8)$ is rational over $k(w_1, w_2, w_3)^{<\tau>}$. In fact, define $u_i = y_i + y_{i+4}$ and $v_i = y_i - y_{i+4}$ for $1 \leq i \leq 4$. Then $k(y_i : 1 \leq i \leq 8) = k(u_i, v_i : 1 \leq i \leq 4)$ and $\tau : u_1 \mapsto u_2 \mapsto u_3 \mapsto u_4 \mapsto u_1$, $\tau : v_1 \mapsto v_2 \mapsto v_3 \mapsto v_4 \mapsto -v_1$. By Theorem 2.2, $k(u_i, v_i : 1 \leq i \leq 4)^{<\tau>} = k(v_i : 1 \leq i \leq 4)^{<\tau>}(Y_1, Y_2, Y_3, Y_4)$ for some Y_j where $\tau(Y_j) = Y_j$ with $1 \leq j \leq 4$. Define $t_i = v_i/v_{i+1}$ for $1 \leq i \leq 3$. Then $\tau : t_1 \mapsto t_2 \mapsto t_3 \mapsto -1/(t_1 t_2 t_3)$ (the same as the action on w_1, w_2, w_3 !) and $k(v_i : 1 \leq i \leq 4)^{<\tau>} = k(t_1, t_2, t_3)^{<\tau>}(f)$ for some f with $\tau(f) = f$ by Theorem 2.3. Hence $k(C_8)$ is rational over $k(w_1, w_2, w_3)^{<\tau>}$.

Since $k(w_1, w_2, w_3)^{<\tau>}$ is k -rational by assumption, it is retract k -rational [Sa1, Sa2, Ka7] (it is the occasion of retract rationality that we require the base field k is infinite). Because the retract rationality is preserved among stably isomorphic fields [Sa2, Proposition 3.6; Ka7, Lemma 3.4], we find that $k(C_8)$ is also retract k -rational. By [Sa2, Theorem 4.12; Ka7, Theorem 2.9], we conclude that $k(\zeta_8)$ is cyclic over k . Done.

Now assume that at least one of $-1, 2, -2$ belongs to $(k^\times)^2$. We will prove that $k(w_1, w_2, w_3)^{<\tau>}$ is rational over k .

By [Ka1, Theorem 1.8; Ya, Theorem 3], $k(w_1, w_2, w_3)^{<\tau>}$ is rational if and only if $-1 \in (k^\times)^2$ or $1 \in 4(k^\times)^4$. The latter condition is nothing but the condition “at least one of $-1, 2, -2$ belongs to $(k^\times)^2$ ”. Hence the result. \square

Theorem 2.7 (Hajja [Ha]) *Let G be a finite group acting on the rational function field $k(x, y)$ be monomial k -automorphisms. Then $k(x, y)^G$ is rational over k .*

Theorem 2.8 (Kang and Plans [KP, Theorem 1.3]) *Let k be any field, G_1 and G_2 two finite groups. If both $k(G_1)$ and $k(G_2)$ are rational over k , then so is $k(G_1 \times G_2)$ over k .*

§3. The proof of Theorem 1.4

We will prove Theorem 1.4 in this section.

By the discussion of Section 1, it suffices to consider those groups G in Theorem 2.1 (with $n \geq 5$) under the assumptions of (1.1), i.e. $\text{char } k \neq 2$ and $\zeta_{2^{n-3}} \in k$. These assumptions will remain in force throughout this section.

Write $\zeta = \zeta_{2^{n-3}} \in k$ from now on. Since $n \geq 5$, $\zeta^{2^{n-5}} \in k$ and $\zeta^{2^{n-5}}$ is a primitive 4-th root of unity. We write $\zeta^{2^{n-5}} = \sqrt{-1}$.

Case 1. $G = G_1$ where G_1 is the group in Theorem 2.1.

G is a metacyclic group. But we cannot apply Theorem 1.7 because $\zeta_{2^{n-2}} \notin k$.

Let V be a k -vector space whose dual space V^* is defined as $V^* = \bigoplus_{g \in G} k \cdot x(g)$ and $h \cdot x(g) = x(hg)$ for any $g, h \in G$. Note that $k(G) = k(x(g) : g \in G)^G = k(V)^G$. We will find a faithful G -subspace W of V^* .

Note that $\langle \sigma^2, \tau \rangle$ is an abelian subgroup of G and $\text{ord}(\sigma^2) = 2^{n-3}$. Define

$$(3.1) \quad \begin{aligned} X &= \sum_{0 \leq i \leq 2^{n-3}-1} \zeta^{-i} [x(\sigma^{2i}) + x(\sigma^{2i}\tau) + x(\sigma^{2i}\tau^2) + x(\sigma^{2i}\tau^3)], \\ Y &= \sum_{\substack{0 \leq i \leq 2^{n-3}-1 \\ 0 \leq j \leq 3}} (\sqrt{-1})^{-j} x(\sigma^{2i}\tau^j). \end{aligned}$$

We find that

$$\begin{aligned} \sigma^2 : X &\mapsto \zeta X, \quad Y \mapsto Y, \\ \tau : X &\mapsto X, \quad Y \mapsto \sqrt{-1}Y. \end{aligned}$$

Define $x_0 = X$, $x_1 = \sigma X$, $y_0 = Y$, $y_1 = \sigma Y$. The actions of σ, τ are given by

$$\begin{aligned} \sigma : x_0 &\mapsto x_1 \mapsto \zeta x_0, \quad y_0 \mapsto y_1 \mapsto y_0, \\ \tau : x_0 &\mapsto x_0, \quad x_1 \mapsto -x_1, \quad y_0 \mapsto \sqrt{-1}y_0, \quad y_1 \mapsto \sqrt{-1}y_1. \end{aligned}$$

It follows that $W = k \cdot x_0 \oplus k \cdot x_1 \oplus k \cdot y_0 \oplus k \cdot y_1$ is a faithful G -subspace of V^* . By Theorem 2.2, $k(G)$ is rational over $k(x_0, x_1, y_0, y_1)^G$. It remains to show that $k(x_0, x_1, y_0, y_1)^{(\sigma, \tau)}$ is rational over k .

Define $z_1 = x_1/x_0$, $z_2 = y_1/y_0$. Then $k(x_0, x_1, y_0, y_1) = k(z_1, z_2, x_0, y_0)$ and

$$\begin{aligned} \sigma : x_0 &\mapsto z_1 x_0, \quad y_0 \mapsto z_2 y_0, \quad z_1 \mapsto \zeta/z_1, \quad z_2 \mapsto 1/z_2, \\ \tau : x_0 &\mapsto x_0, \quad y_0 \mapsto \sqrt{-1}y_0, \quad z_1 \mapsto -z_1, \quad z_2 \mapsto z_2. \end{aligned}$$

By Theorem 2.3, $k(z_1, z_2, x_0, y_0)^{\langle \sigma, \tau \rangle} = k(z_1, z_2)^{\langle \sigma, \tau \rangle}(z_3, z_4)$ for some z_3, z_4 with $\sigma(z_j) = \tau(z_j) = z_j$ for $j = 3, 4$.

The actions of σ and τ on z_1, z_2 are monomial automorphisms. By Theorem 2.7, $k(z_1, z_2)^{\langle \sigma, \tau \rangle}$ is rational. Thus $k(x_0, x_1, y_0, y_1)^{\langle \sigma, \tau \rangle}$ is also rational over k .

Case 2. $G = G_2, G_3, G_{10}$ or G_{11} .

These four groups are direct products of subgroups $\langle \sigma, \tau \rangle$ and $\langle \lambda \rangle$. We may apply Theorem 1.8 to study $k(G)$ since $H := \langle \sigma, \tau \rangle$ is a group of order 2^{n-1} , $\text{ord}(\sigma) = 2^{n-2}$ and $\zeta_{2^{n-3}} \in k$. By Theorem 1.3 we find that $k(H)$ is rational over k .

Case 3. $G = G_4$.

As in the proof of Case 1. $G = G_1$, we will find a faithful G -subspace W in $V^* = \bigoplus_{g \in G} k \cdot x(g)$. The construction of W is similar to that in Case 1, but some modification should be made.

Although $\langle \sigma^2, \tau \rangle$ is an abelian subgroup of G , we will consider $\langle \sigma^2 \rangle$ instead. Explicitly, define

$$X = \sum_{0 \leq i \leq 2^{n-3}-1} \zeta^{-i} [x(\sigma^{2i}) + x(\sigma^{2i}\tau)].$$

It follows that $\sigma^2(X) = \zeta X$ and $\tau(X) = X$.

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \lambda X$, $x_3 = \lambda \sigma X$. We find that

$$\begin{aligned} \sigma : x_0 &\mapsto x_1 \mapsto \zeta x_0, \quad x_2 \mapsto x_3 \mapsto \zeta x_2, \\ \tau : x_0 &\mapsto x_0, \quad x_1 \mapsto x_1, \quad x_2 \mapsto -x_2, \quad x_3 \mapsto -x_3, \\ \lambda : x_0 &\mapsto x_2 \mapsto x_0, \quad x_1 \mapsto x_3 \mapsto x_1. \end{aligned}$$

Note that G acts faithfully on $k(x_i : 0 \leq i \leq 3)$. Hence $k(G)$ is rational over $k(x_i : 0 \leq i \leq 3)^G$ by Theorem 2.2.

Define $y_0 = x_0^{2^{n-3}}$, $y_1 = x_1/x_0$, $y_2 = x_2/x_1$, $y_3 = x_3/x_2$. Then $k(x_i : 0 \leq i \leq 3)^{\langle \sigma^2 \rangle} = k(y_i : 0 \leq i \leq 3)$ and

$$\begin{aligned} \sigma : y_0 &\mapsto y_1^{2^{n-3}} y_0, \quad y_1 \mapsto \zeta/y_1, \quad y_2 \mapsto \zeta^{-1} y_1 y_2 y_3, \quad y_3 \mapsto \zeta/y_3, \\ \tau : y_0 &\mapsto y_0, \quad y_1 \mapsto y_1, \quad y_2 \mapsto -y_2, \quad y_3 \mapsto y_3, \\ \lambda : y_0 &\mapsto y_1^{2^{n-3}} y_2^{2^{n-3}} y_0, \quad y_1 \mapsto y_3 \mapsto y_1, \quad y_2 \mapsto 1/(y_1 y_2 y_3). \end{aligned}$$

By Theorem 2.3, we find that $k(y_i : 0 \leq i \leq 3)^{\langle \sigma, \tau, \lambda \rangle} = k(y_i : 1 \leq i \leq 3)^{\langle \sigma, \tau, \lambda \rangle}(y_4)$ for some y_4 with $\sigma(y_4) = \tau(y_4) = \lambda(y_4) = y_4$.

It is clear that $k(y_i : 1 \leq i \leq 3)^{\langle \tau \rangle} = k(y_1, y_2^2, y_3)$.

Define $z_1 = y_1$, $z_2 = y_3$, $z_3 = y_1 y_3 y_2^2$. Then $k(y_1, y_2^2, y_3) = k(z_i : 1 \leq i \leq 3)$ and

$$\begin{aligned} \sigma : z_1 &\mapsto \zeta/z_1, \quad z_2 \mapsto \zeta/z_2, \quad z_3 \mapsto z_3, \\ \lambda : z_1 &\mapsto z_2 \mapsto z_1, \quad z_3 \mapsto 1/z_3. \end{aligned}$$

By Theorem 2.4, $k(z_i : 1 \leq i \leq 3)^{\langle \sigma, \lambda \rangle}$ is rational over k .

Case 4. $G = G_5$.

The proof is similar to Case 3. $G = G_4$. We define X such that $\sigma^2(X) = \zeta X$, $\lambda(X) = X$ (note that in the present case we require $\lambda(X) = X$ instead of $\tau(X) = X$).

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \tau X$, $x_3 = \tau \sigma X$. It follows that

$$\begin{aligned}\sigma : x_0 &\mapsto x_1 \mapsto \zeta x_0, \quad x_2 \mapsto x_3 \mapsto \zeta x_2, \\ \tau : x_0 &\mapsto x_2 \mapsto x_0, \quad x_1 \mapsto x_3 \mapsto x_1, \\ \lambda : x_0 &\mapsto x_0, \quad x_1 \mapsto x_3 \mapsto x_1, \quad x_2 \mapsto x_2.\end{aligned}$$

It follows that G acts faithfully on $k(x_i : 0 \leq i \leq 3)$. By Theorem 2.2 it suffices to show that $k(x_i : 0 \leq i \leq 3)^G$ is rational over k .

Define $y_0 = x_0 - x_2$, $y_1 = x_1 - x_3$, $y_2 = x_0 + x_2$, $y_3 = x_1 + x_3$. It follows that $k(x_i : 0 \leq i \leq 3) = k(y_0 : 0 \leq i \leq 3)$ and

$$\begin{aligned}\sigma : y_0 &\mapsto y_1 \mapsto \zeta y_0, \quad y_2 \mapsto y_3 \mapsto \zeta y_2, \\ \tau : y_0 &\mapsto -y_0, \quad y_1 \mapsto -y_1, \quad y_2 \mapsto y_2, \quad y_3 \mapsto y_3, \\ \lambda : y_0 &\mapsto y_0, \quad y_1 \mapsto -y_1, \quad y_2 \mapsto y_2, \quad y_3 \mapsto y_3.\end{aligned}$$

By Theorem 2.2 $k(y_i : 0 \leq i \leq 3)^G = k(y_0, y_1)^G(y_4, y_5)$ for some y_4, y_5 with $g(y_4) = y_4$, $g(y_5) = y_5$ for any $g \in G$. Note the the actions of G on y_0, y_1 are monomial automorphisms. By Theorem 2.7 $k(y_0, y_1)^G$ is rational over k .

Case 5. $G = G_6, G_7$.

Consider the case $G = G_6$ first.

Note that $\langle \sigma^2, \tau^2 \rangle$ is an abelian subgroup of G . As in the proof of Case 1. $G = G_1$ we define X and Y in $V^* = \bigoplus_{g \in G} k \cdot x(g)$ by

$$\begin{aligned}X &= \sum_{0 \leq i \leq 2^{n-3}-1} \zeta^{-i} [x(\sigma^{2i}) + x(\sigma^{2i}\tau^2)], \\ Y &= \sum_{\substack{0 \leq i \leq 2^{n-3}-1 \\ 0 \leq j \leq 3}} (\sqrt{-1})^{-j} x(\sigma^{2i}\tau^j).\end{aligned}\tag{3.2}$$

It follows that $\sigma^2(X) = \zeta X$, $\tau^2(X) = X$, $\sigma^2(Y) = Y$, $\tau(Y) = \sqrt{-1}Y$.

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \tau X$, $x_3 = \tau \sigma X$, $y_0 = Y$, $y_1 = \sigma Y$. We get

$$\begin{aligned}\sigma : x_0 &\mapsto x_1 \mapsto \zeta x_0, \quad x_2 \mapsto \zeta^{-1} x_3, \quad x_3 \mapsto x_2, \quad y_0 \mapsto y_1 \mapsto y_0, \\ \tau : x_0 &\mapsto x_2 \mapsto x_0, \quad x_1 \mapsto x_3 \mapsto x_1, \quad y_0 \mapsto \sqrt{-1} y_0, \quad y_1 \mapsto \sqrt{-1} y_1.\end{aligned}$$

Note that G acts faithfully on $k(x_i, y_0, y_1 : 0 \leq i \leq 3)$. We will show that $k(x_i, y_0, y_1 : 0 \leq i \leq 3)^{\langle \sigma, \tau \rangle}$ is rational over k .

Define $y_2 = y_1/y_0$. It follows that $\sigma(y_2) = 1/y_2$, $\sigma(y_0) = y_2 y_0$, $\tau(y_2) = y_2$, $\tau(y_0) = \sqrt{-1}y_0$. By Theorem 2.3 $k(x_i, y_0, y_1 : 0 \leq i \leq 3)^{\langle \sigma, \tau \rangle} = k(x_i, y_2, y_0 : 0 \leq i \leq 3)^{\langle \sigma, \tau \rangle} = k(x_i, y_2 : 0 \leq i \leq 3)^{\langle \sigma, \tau \rangle}(y_3)$ for some y_3 with $\sigma(y_3) = \tau(y_3) = y_3$.

Define $y_4 = (1 - y_2)/(1 + y_2)$. Then $\sigma(y_4) = -y_4$, $\tau(y_4) = y_4$. By Theorem 2.3 $k(x_i, y_2 : 0 \leq i \leq 3)^{\langle \sigma, \tau \rangle} = k(x_i : 0 \leq i \leq 3)^{\langle \sigma, \tau \rangle}(y_5)$ for some y_5 with $\sigma(y_5) = \tau(y_5) = y_5$.

Define $z_0 = x_0$, $z_1 = x_1/x_0$, $z_2 = x_3/x_2$, $z_3 = x_2/x_1$. We find that

$$\begin{aligned}\sigma : z_0 &\mapsto z_1 z_0, \quad z_1 \mapsto \zeta/z_1, \quad z_2 \mapsto \zeta/z_2, \quad z_3 \mapsto \zeta^{-2} z_1 z_2 z_3, \\ \tau : z_0 &\mapsto z_1 z_3 z_0, \quad z_1 \mapsto z_2 \mapsto z_1, \quad z_3 \mapsto 1/(z_1 z_2 z_3).\end{aligned}$$

By Theorem 2.3 $k(x_i : 0 \leq i \leq 3)^{\langle \sigma, \tau \rangle} = k(z_i : 0 \leq i \leq 3)^{\langle \sigma, \tau \rangle} = k(z_i : 1 \leq i \leq 3)^{\langle \sigma, \tau \rangle}(z_4)$ for some z_4 with $\sigma(z_4) = \tau(z_4) = z_4$.

Define $u_1 = z_3^{2^{n-4}}$. Then $k(z_i : 1 \leq i \leq 3)^{\langle \sigma^2 \rangle} = k(z_1, z_2, u_1)$ and

$$\begin{aligned}\sigma : z_1 &\mapsto \zeta/z_1, \quad z_2 \mapsto \zeta/z_2, \quad u_1 \mapsto (z_1 z_2)^{2^{n-4}} u_1, \\ \tau : z_1 &\mapsto z_2 \mapsto z_1, \quad u_1 \mapsto ((z_1 z_2)^{2^{n-4}} \cdot u_1)^{-1}.\end{aligned}$$

Define $u_2 = (z_1 z_2)^{2^{n-5}} u_1$. Then $k(z_1, z_2, u_1) = k(z_1, z_2, u_2)$ and

$$\begin{aligned}\sigma : z_1 &\mapsto \zeta/z_1, \quad z_2 \mapsto \zeta/z_2, \quad u_2 \mapsto -u_2, \\ \tau : z_1 &\mapsto z_2 \mapsto z_1, \quad u_2 \mapsto 1/u_2.\end{aligned}$$

By Theorem 2.4 $k(z_1, z_2, u_2)^{\langle \sigma, \tau \rangle}$ is rational over k . This solves the case $G = G_6$.

When $G = G_7$, we use the same X and Y in (3.2). Define $x_0, x_1, x_2, x_3, y_0, y_1$ by the same formula. The proof is almost the same as $G = G_6$. Done.

Case 6. $G = G_8$.

Note that $\tau^8 = 1$ and $\sigma\tau^2 = \tau^2\sigma$.

Define X and Y in $V^* = \bigoplus_{g \in G} k \cdot x(g)$ by

$$X = \sum_{\substack{0 \leq i \leq 2^{n-3}-1 \\ 0 \leq j \leq 3}} \zeta^{-i} x(\sigma^{2i} \tau^{2j}), \quad Y = \sum_{\substack{0 \leq i \leq 2^{n-3}-1 \\ 0 \leq j \leq 3}} (\sqrt{-1})^{-j} x(\sigma^{2i} \tau^{2j}).$$

It follows that $\sigma^2(X) = \zeta X$, $\sigma^2(Y) = Y$, $\tau^2(X) = X$, $\tau^2(Y) = \sqrt{-1}Y$.

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \tau X$, $x_3 = \tau\sigma X$, $y_0 = Y$, $y_1 = \sigma Y$, $y_2 = \tau Y$, $y_3 = \tau\sigma Y$. We find that

$$\begin{aligned}\sigma : x_0 &\mapsto x_1 \mapsto \zeta x_0, \quad x_2 \mapsto \zeta^{-1} x_3, \quad x_3 \mapsto x_2, \quad y_0 \leftrightarrow y_1, \quad y_2 \leftrightarrow y_3, \\ \tau : x_0 &\leftrightarrow x_2, \quad x_1 \leftrightarrow x_3, \quad y_0 \mapsto y_2 \mapsto \sqrt{-1} y_0, \quad y_1 \mapsto y_3 \mapsto \sqrt{-1} y_1.\end{aligned}$$

Since $G = \langle \sigma, \tau \rangle$ acts faithfully on $k(x_i, y_i : 0 \leq i \leq 3)$, it remains to show that $k(x_i, y_i : 0 \leq i \leq 3)^{\langle \sigma, \tau \rangle}$ is rational over k .

Define $z_i = x_i y_i$ for $0 \leq i \leq 3$. We get

$$(3.3) \quad \begin{aligned} \sigma : z_0 &\mapsto z_1 \mapsto \zeta z_0, \quad z_2 \mapsto \zeta^{-1} z_3, \quad z_3 \mapsto z_2, \\ \tau : z_0 &\mapsto z_2 \mapsto \sqrt{-1} z_0, \quad z_1 \mapsto z_3 \mapsto \sqrt{-1} z_1. \end{aligned}$$

Note that $k(x_i, y_i : 0 \leq i \leq 3) = k(x_i, z_i : 0 \leq i \leq 3)$ and G acts faithfully on $k(z_i : 0 \leq i \leq 3)$. By Theorem 2.2 $k(x_i, z_i : 0 \leq i \leq 3)^{\langle \sigma, \tau \rangle} = k(z_i : 0 \leq i \leq 3)^{\langle \sigma, \tau \rangle} (X_0, X_1, X_2, X_3)$ for some X_i ($0 \leq i \leq 3$) with $\sigma(X_i) = \tau(X_i) = X_i$.

Define $u_0 = z_0$, $u_1 = z_1/z_0$, $u_2 = z_3/z_2$, $u_3 = z_2/z_1$. The actions are given by

$$(3.4) \quad \begin{aligned} \sigma : u_0 &\mapsto u_1 u_0, \quad u_1 \mapsto \zeta/u_1, \quad u_2 \mapsto \zeta/u_2, \quad u_3 \mapsto \zeta^{-2} u_1 u_2 u_3, \\ \tau : u_0 &\mapsto u_1 u_3 u_0, \quad u_1 \leftrightarrow u_2, \quad u_3 \mapsto \sqrt{-1}/(u_1 u_2 u_3). \end{aligned}$$

By Theorem 2.3 $k(z_i : 0 \leq i \leq 3)^{\langle \sigma, \tau \rangle} = k(u_i : 0 \leq i \leq 3)^{\langle \sigma, \tau \rangle} = k(u_i : 1 \leq i \leq 3)^{\langle \sigma, \tau \rangle} (u_4)$ for some u_4 with $\sigma(u_4) = \tau(u_4) = u_4$.

Define $v_1 = u_3^{2^{n-4}}$. Then $k(u_i : 1 \leq i \leq 3)^{\langle \sigma^2 \rangle} = k(u_1, u_2, v_1)$ and $\sigma(v_1) = (u_1 u_2)^{2^{n-4}} v_1$, $\tau(v_1) = \varepsilon / ((u_1 u_2)^{2^{n-4}} u_4)$ where $\varepsilon = 1$ if $n \geq 6$, and $\varepsilon = -1$ if $n = 5$.

Define $v_2 = (u_1 u_2)^{2^{n-5}} v_1$. Then $\sigma(v_2) = -v_2$, $\tau(v_2) = \varepsilon/v_2$. Since $k(u_1, u_2, v_1)^{\langle \sigma, \tau \rangle} = k(u_1, u_2, v_2)^{\langle \sigma, \tau \rangle}$ is rational over k by Theorem 2.4, the proof is finished.

Case 7. $G = G_9$.

Note that $\sigma^2 \tau = \tau \sigma^2$.

Define X and Y in $V^* = \bigoplus_{g \in G} k \cdot x(g)$ by

$$X = \sum_{\substack{0 \leq i \leq 2^{n-3}-1 \\ 0 \leq j \leq 3}} \zeta^{-i} x(\sigma^{2i} \tau^j), \quad Y = \sum_{\substack{0 \leq i \leq 2^{n-3}-1 \\ 0 \leq j \leq 3}} (\sqrt{-1})^{-j} x(\sigma^{2i} \tau^j).$$

It follows that $\sigma^2(X) = \zeta X$, $\sigma^2(Y) = Y$, $\tau(X) = X$, $\tau(Y) = \sqrt{-1}Y$.

Define $x_0 = X$, $x_1 = \sigma X$, $y_0 = Y$, $y_1 = \sigma Y$. We get

$$\begin{aligned} \sigma : x_0 &\mapsto x_1 \mapsto \zeta x_0, \quad y_0 \mapsto y_1 \mapsto y_0, \\ \tau : x_0 &\mapsto x_0, \quad x_1 \mapsto x_1, \quad y_0 \mapsto \sqrt{-1} y_0, \quad y_1 \mapsto -\sqrt{-1} y_1. \end{aligned}$$

It remains to prove $k(x_0, x_1, y_0, y_1)^{\langle \sigma, \tau \rangle}$ is rational over k . The proof is almost the same as Case 1. $G = G_1$. Done.

Case 8. $G = G_{12}$.

Define $X \in V^* = \bigoplus_{g \in G} h \cdot x(g)$ by

$$X = \sum_{0 \leq i \leq 2^{n-3}-1} \zeta^{-i} [x(\sigma^{2i}) + x(\sigma^{2i} \tau)].$$

Then $\sigma^2 X = \zeta X$, $\tau X = X$.

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \lambda X$, $x_3 = \lambda \sigma X$. We find that

$$\begin{aligned}\sigma : x_0 &\mapsto x_1 \mapsto \zeta x_0, \quad x_2 \mapsto \zeta^{-1} x_3, \quad x_3 \mapsto x_2, \\ \tau : x_0 &\mapsto x_0, \quad x_1 \mapsto x_1, \quad x_2 \mapsto -x_2, \quad x_3 \mapsto -x_3, \\ \lambda : x_0 &\leftrightarrow x_2, \quad x_1 \leftrightarrow x_3.\end{aligned}$$

Since $G = \langle \sigma, \tau, \lambda \rangle$ is faithful on $k(x_i : 0 \leq i \leq 3)$, it remains to show that $k(x_i : 0 \leq i \leq 3)^{\langle \sigma, \tau, \lambda \rangle}$ is rational over k .

Define $y_0 = x_0$, $y_1 = x_1/x_0$, $y_2 = x_3/x_2$, $y_3 = x_2/x_1$. We get

$$\begin{aligned}(3.5) \quad \sigma : y_0 &\mapsto y_1 y_0, \quad y_1 \mapsto \zeta/y_1, \quad y_2 \mapsto \zeta/y_2, \quad y_3 \mapsto \zeta^{-2} y_1 y_2 y_3, \\ \tau : y_0 &\mapsto y_0, \quad y_1 \mapsto y_1, \quad y_2 \mapsto y_2, \quad y_3 \mapsto -y_3, \\ \lambda : y_0 &\mapsto y_1 y_3 y_0, \quad y_1 \leftrightarrow y_2, \quad y_3 \mapsto 1/(y_1 y_2 y_3).\end{aligned}$$

By Theorem 2.3 $k(y_i : 0 \leq i \leq 3)^{\langle \sigma, \tau, \lambda \rangle} = k(y_i : 1 \leq i \leq 3)^{\langle \sigma, \tau, \lambda \rangle}(y_4)$ for some y_4 with $\sigma(y_4) = \tau(y_4) = \lambda(y_4) = y_4$.

Define $z_1 = y_3^2$. Then $k(y_i : 1 \leq i \leq 3)^{\langle \tau \rangle} = k(y_1, y_2, z_1)$ and $\sigma(z_1) = \zeta^{-4} y_1^2 y_2^2 z_1$, $\lambda(z_1) = 1/(y_1^2 y_2^2 z_1)$.

Define $z_2 = z_1^{2^{n-5}}$. Then $k(y_1, y_2, z_1)^{\langle \sigma^2 \rangle} = k(y_1, y_2, z_2)$ and $\sigma(z_2) = (y_1 y_2)^{2^{n-4}} z_2$, $\lambda(z_2) = 1/((y_1 y_2)^{2^{n-4}} z_2)$.

Define $z_3 = (y_1 y_2)^{2^{n-5}} z_2$. We find that $k(y_1, y_2, z_2) = k(y_1, y_2, z_3)$ and $\sigma(z_3) = -z_3$, $\lambda(z_3) = 1/z_3$. By Theorem 2.4, $k(y_1, y_2, y_3)^{\langle \sigma, \tau \rangle}$ is rational over k . Done.

Case 9. $G = G_{13}, G_{14}$.

We consider the case $G = G_{13}$ only, because the proof for $G = G_{14}$ is almost the same (with the same way of changing the variables).

Define X and Y in $V^* = \bigoplus_{g \in G} k \cdot x(g)$ by

$$\begin{aligned}X &= \sum_{0 \leq i \leq 2^{n-3}-1} \zeta^{-i} [x(\sigma^{2i}) + x(\sigma^{2i} \tau)], \\ Y &= \sum_{0 \leq i \leq 2^{n-3}-1} x(\sigma^{2i}) - \sum_{0 \leq i \leq 2^{n-3}-1} x(\sigma^{2i} \tau).\end{aligned}$$

We find that $\sigma^2(X) = \zeta X$, $\sigma^2(Y) = Y$, $\tau(X) = X$, $\tau(Y) = -Y$.

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \lambda X$, $x_3 = \lambda \sigma X$, $y_0 = Y$, $y_1 = \sigma Y$, $y_2 = \lambda Y$, $y_3 = \lambda \sigma Y$. It follows that

$$\begin{aligned}\sigma : x_0 &\mapsto x_1 \mapsto \zeta x_0, \quad x_2 \mapsto \zeta^{-1} x_3, \quad x_3 \mapsto x_2, \quad y_0 \leftrightarrow y_1, \quad y_2 \leftrightarrow -y_3, \\ \tau : x_i &\mapsto x_i, \quad y_i \mapsto -y_i, \\ \lambda : x_0 &\leftrightarrow x_2, \quad x_1 \leftrightarrow x_3, \quad y_0 \leftrightarrow y_2, \quad y_1 \leftrightarrow y_3.\end{aligned}$$

Note that G acts faithfully on $k(x_i, y_i : 0 \leq i \leq 3)$. Thus it remains to show that $k(x_i, y_i : 0 \leq i \leq 3)^{\langle \sigma, \tau, \lambda \rangle}$ is rational over k .

Define $x_4 = y_0 + y_1, x_5 = y_2 + y_3, x_6 = y_0 - y_1, x_7 = y_2 - y_3$. Then $k(x_i, y_i : 0 \leq i \leq 3) = k(x_i : 0 \leq i \leq 7)$, and $\sigma(x_i) = x_i$ for $i = 4, 7$, $\sigma(x_i) = -x_i$ for $i = 5, 6$, $\tau(x_i) = -x_i$ for $4 \leq i \leq 7$, $\lambda : x_4 \leftrightarrow x_5, x_6 \leftrightarrow x_7$.

Apply Theorem 2.2 to $k(x_i : 0 \leq i \leq 7)$. It suffices to prove that $k(x_i : 0 \leq i \leq 5)^{\langle \sigma, \tau, \lambda \rangle}$ is rational over k .

Define $Z = x_5/x_4$. Then $k(x_i : 0 \leq i \leq 5) = k(x_i, Z : 0 \leq i \leq 4)$ and $\sigma(Z) = -Z, \tau(Z) = Z, \lambda(Z) = 1/Z$. Apply Theorem 2.3 to $k(x_i : 0 \leq i \leq 5)$. It remains to prove that $k(x_i, Z : 0 \leq i \leq 3)^{\langle \sigma, \tau, \lambda \rangle}$ is rational over k . Note that the action of τ becomes trivial on $k(x_i, Z : 0 \leq i \leq 3)$.

Define $u_0 = x_0, u_1 = x_1/x_0, u_2 = x_3/x_2, u_3 = x_2/x_1, u_4 = Z$. By Theorem 2.3 $k(x_i, Z : 0 \leq i \leq 3)^{\langle \sigma, \lambda \rangle} = k(u_i : 1 \leq i \leq 4)^{\langle \sigma, \lambda \rangle}(U)$ for some element U fixed by the action of G . The actions of σ and λ are given by

$$\begin{aligned} \sigma : u_1 &\mapsto \zeta/u_1, \quad u_2 \mapsto \zeta/u_2, \quad u_3 \mapsto \zeta^{-2}u_1u_2u_3, \quad u_4 \mapsto -u_4, \\ \lambda : u_1 &\leftrightarrow u_2, \quad u_3 \mapsto 1/(u_1u_2u_3), \quad u_4 \mapsto 1/u_4. \end{aligned}$$

Note that σ^2 fixes u_1, u_2, u_4 and $\sigma^2(u_3) = \zeta^{-2}u_3$. Define $u_5 = u_3^{2^{n-4}}$. Then $k(u_i : 1 \leq i \leq 4)^{\langle \sigma^2 \rangle} = k(u_1, u_2, u_4, u_5)$ and $\sigma(u_5) = (u_1u_2)^{2^{n-4}}u_5, \lambda(u_5) = 1/((u_1u_2)^{2^{n-4}}u_5)$.

Define $u_6 = (u_1u_2)^{2^{n-5}}u_5$. Then $k(u_1, u_2, u_4, u_5) = k(u_1, u_2, u_4, u_6)$ and we get

$$\begin{aligned} \sigma : u_1 &\mapsto \zeta/u_1, \quad u_2 \mapsto \zeta/u_2, \quad u_6 \mapsto -u_6, \quad u_4 \mapsto -u_4, \\ \lambda : u_1 &\leftrightarrow u_2, \quad u_6 \mapsto 1/u_6, \quad u_4 \mapsto 1/u_4. \end{aligned}$$

Define $u_7 = u_4u_6$. Then $\sigma(u_7) = u_7, \lambda(u_7) = 1/u_7$. Define $u_8 = (1 - u_7)/(1 + u_7)$. Then $\sigma(u_8) = u_8, \lambda(u_8) = -u_8$. Since $k(u_1, u_2, u_4, u_6) = k(u_1, u_2, u_6, u_8)$, we may apply Theorem 2.3. Thus it suffices to prove that $k(u_1, u_2, u_6)^{\langle \sigma, \lambda \rangle}$ is rational over k . By Theorem 2.4 $k(u_1, u_2, u_6)^{\langle \sigma, \lambda \rangle}$ is rational over k . Done.

Case 10. $G = G_{15}, G_{16}, G_{17}, G_{18}, G_{24}, G_{25}$.

These cases were proved in [Ka6, Section 5]. Note that in Cases 5 ~ 8 of [Ka6, Section 5], only $\zeta_{2^{n-3}} \in k$ was used. Hence the result.

Case 11. $G = G_{19}, G_{20}$.

We consider the case $G = G_{19}$ only, because the proof for $G = G_{20}$ is almost the same.

Define $X \in V^* = \bigoplus_{g \in G} k \cdot x(g)$ by

$$X = \sum_{0 \leq i \leq 2^{n-3}-1} \zeta^{-i} [x(\sigma^{2i}) + x(\sigma^{2i}\tau^2)].$$

Then $\sigma^2(X) = \zeta X$ and $\tau^2(X) = X$.

Define $x_0 = X, x_1 = \sigma X, x_2 = \tau X, x_3 = \tau\sigma X$. We find that

$$\begin{aligned} \sigma : x_0 &\mapsto x_1 \mapsto \zeta x_0, \quad x_2 \mapsto \sqrt{-1}x_3, \quad x_3 \mapsto \sqrt{-1}\zeta x_2, \\ \tau : x_0 &\leftrightarrow x_2, \quad x_1 \mapsto x_3 \mapsto -x_1. \end{aligned}$$

Thus G acts faithfully on $k(x_i : 0 \leq i \leq 3)$. It remains to prove $k(x_i : 0 \leq i \leq 3)^{(\sigma, \tau)}$ is rational over k .

Define $u_0 = x_0$, $u_1 = x_1/x_0$, $u_2 = x_3/x_2$, $u_3 = x_2/x_1$. We find that

$$(3.6) \quad \begin{aligned} \sigma : u_0 &\mapsto u_1 u_0, \quad u_1 \mapsto \zeta/u_1, \quad u_2 \mapsto \zeta/u_2, \quad u_3 \mapsto \sqrt{-1}\zeta^{-1}u_1 u_2 u_3, \\ \tau : u_0 &\mapsto u_1 u_3 u_0, \quad u_1 \mapsto u_2 \mapsto -u_1, \quad u_3 \mapsto 1/(u_1 u_2 u_3). \end{aligned}$$

Compare the formula (3.6) with the formula (3.4) in the proof of Case 6. $G = G_8$. It is not difficult to see that the proof is almost the same as that of Case 6. $G = G_8$ (by taking the fixed field of the subgroup $\langle \sigma^2 \rangle$ first, and then making similar changes of variables). Done.

Case 12. $G = G_{21}$.

Note that $\tau^8 = 1$ and $\sigma^2 \tau = \tau \sigma^2$.

Define X and Y in $V^* = \bigoplus_{g \in G} k \cdot x(g)$ by

$$X = \sum_{\substack{0 \leq i \leq 2^{n-3}-1 \\ 0 \leq j \leq 3}} \zeta^{-i} x(\sigma^{2i} \tau^{2j}), \quad Y = \sum_{\substack{0 \leq i \leq 2^{n-3}-1 \\ 0 \leq j \leq 2}} (\sqrt{-1})^{-j} x(\sigma^{2i} \tau^{2j}).$$

Then $\sigma^2(X) = \zeta X$, $\sigma^2(Y) = Y$, $\tau^2(X) = X$, $\tau^2(Y) = \sqrt{-1}Y$.

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \tau X$, $x_3 = \tau \sigma X$, $y_0 = Y$, $y_1 = \sigma Y$, $y_2 = \tau Y$, $y_3 = \tau \sigma Y$. We find that

$$\begin{aligned} \sigma : x_0 &\mapsto x_1 \mapsto \zeta x_0, \quad x_2 \mapsto x_3 \mapsto \zeta x_2, \quad y_0 \leftrightarrow y_1, \quad y_2 \leftrightarrow \sqrt{-1}y_3, \\ \tau : x_0 &\leftrightarrow x_2, \quad x_1 \leftrightarrow x_3, \quad y_0 \mapsto y_2 \mapsto \sqrt{-1}y_0, \quad y_1 \mapsto y_3 \mapsto -\sqrt{-1}y_1. \end{aligned}$$

Since G is faithful on $k(x_i, y_i : 0 \leq i \leq 3)$, it remains to show that $k(x_i, y_i : 0 \leq i \leq 3)^{(\sigma, \tau)}$ is rational over k .

Define $z_i = x_i y_i$ for $0 \leq i \leq 3$. It follows that

$$(3.7) \quad \begin{aligned} \sigma : z_0 &\mapsto z_1 \mapsto \zeta z_0, \quad z_2 \mapsto \sqrt{-1}z_3, \quad z_3 \mapsto -\sqrt{-1}\zeta z_2, \\ \tau : z_0 &\mapsto z_2 \mapsto \sqrt{-1}z_0, \quad z_1 \mapsto z_3 \mapsto -\sqrt{-1}z_1. \end{aligned}$$

Compare the formulae (3.7) and (3.3). They are almost the same. Thus it is obvious that $k(x_i, y_i : 0 \leq i \leq 3)^{(\sigma, \tau)}$ is rational over k .

Case 13. $G = G_{22}, G_{23}$.

We consider the case $G = G_{23}$, because the proof for $G = G_{22}$ is almost the same.

Define $X \in V^* = \bigoplus_{g \in G} k \cdot x(g)$ by

$$X = \sum_{0 \leq i \leq 2^{n-3}-1} \zeta^{-i} [x(\sigma^{2i}) + x(\sigma^{2i} \tau)].$$

Then $\sigma^2(X) = \zeta X$, $\tau(X) = X$.

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \lambda X$, $x_3 = \lambda \sigma X$. We find that

$$\begin{aligned}\sigma : x_0 &\mapsto x_1 \mapsto \zeta x_0, \quad x_2 \mapsto \sqrt{-1}\zeta^{-1}x_3, \quad x_3 \mapsto \sqrt{-1}x_2, \\ \tau : x_0 &\mapsto x_0, x_1 \mapsto x_1, \quad x_2 \mapsto -x_2, \quad x_3 \mapsto -x_3, \\ \lambda : x_0 &\leftrightarrow x_2, \quad x_1 \leftrightarrow x_3.\end{aligned}$$

Note that G acts faithfully on $k(x_i : 0 \leq i \leq 3)$. It remains to show that $k(x_i : 0 \leq i \leq 3)^{\langle \sigma, \tau, \lambda \rangle}$ is rational over k .

Define $y_0 = x_0$, $y_1 = x_1/x_0$, $y_2 = x_3/x_2$, $y_3 = x_2/x_1$. We get

$$(3.8) \quad \begin{aligned}\sigma : y_0 &\mapsto y_1 y_0, \quad y_1 \mapsto \zeta/y_1, \quad y_2 \mapsto \zeta/y_2, \quad y_3 \mapsto \sqrt{-1}\zeta^{-2}y_1 y_2 y_3, \\ \tau : y_0 &\mapsto y_0, \quad y_1 \mapsto y_1, \quad y_2 \mapsto y_2, \quad y_3 \mapsto -y_3, \\ \lambda : y_0 &\mapsto y_1 y_3 y_0, \quad y_1 \leftrightarrow y_2, \quad y_3 \leftrightarrow 1/(y_1 y_2 y_3).\end{aligned}$$

Compare the formula (3.8) with the formula (3.5) in the proof of Case 8. $G = G_{12}$. It is not difficult to show that $k(x_i : 0 \leq i \leq 3)^{\langle \sigma, \tau, \lambda \rangle}$ is rational over k in the present case.

Case 14. $G = G_{26}$.

Note that $\lambda^4 = 1$ and $\sigma^2 \tau = \tau \sigma^2$.

Define $X \in V^* = \bigoplus_{g \in G} k \cdot x(g)$ by

$$X = \sum_{0 \leq i \leq 3} (\sqrt{-1})^{-i} [x(\sigma^{2i}) + x(\sigma^{2i} \tau)].$$

Then $\sigma^2(X) = \sqrt{-1}X$, $\tau(X) = X$.

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \lambda X$, $x_3 = \lambda \sigma X$. We find that

$$\begin{aligned}\sigma : x_0 &\mapsto x_1 \mapsto \sqrt{-1}x_0, \quad x_2 \mapsto x_3 \mapsto -\sqrt{-1}x_2, \\ \tau : x_0 &\mapsto x_0, \quad x_1 \mapsto -x_1, \quad x_2 \mapsto x_2, \quad x_3 \mapsto -x_3, \\ \lambda : x_0 &\mapsto x_2 \mapsto -x_0, \quad x_1 \mapsto x_3 \mapsto -x_1.\end{aligned}$$

Since G is faithful on $k(x_i : 0 \leq i \leq 3)$, it remains to show that $k(x_i : 0 \leq i \leq 3)^{\langle \sigma, \tau, \lambda \rangle}$ is rational over k .

Define $y_0 = x_0$, $y_1 = x_1/x_0$, $y_2 = x_3/x_2$, $y_3 = x_2/x_1$. We get

$$\begin{aligned}\sigma : y_0 &\mapsto y_1 y_0, \quad y_1 \mapsto \sqrt{-1}/y_1, \quad y_2 \mapsto -\sqrt{-1}/y_2, \quad y_3 \mapsto -\sqrt{-1}y_1 y_2 y_3, \\ \tau : y_0 &\mapsto y_0, \quad y_1 \mapsto -y_1, \quad y_2 \mapsto -y_2, \quad y_3 \mapsto -y_3, \\ \lambda : y_0 &\mapsto y_1 y_3 y_0, \quad y_1 \leftrightarrow y_2, \quad y_3 \mapsto -1/(y_1 y_2 y_3).\end{aligned}$$

By Theorem 2.3 $k(y_i : 0 \leq i \leq 3)^{\langle \sigma, \tau, \lambda \rangle} = k(y_i : 1 \leq i \leq 3)^{\langle \sigma, \tau, \lambda \rangle}(y_4)$ for some y_4 with $\sigma(y_4) = \tau(y_4) = \lambda(y_4) = y_4$.

Define $v_0 = y_3^2$. Then $k(y_i : 1 \leq i \leq 3)^{\langle \sigma^2 \rangle} = k(v_0, y_1, y_2)$ and

$$\sigma(v_0) = -(y_1 y_2)^2 v_0, \quad \tau(v_0) = v_0, \quad \lambda(v_0) = 1/(y_1^2 y_2^2 v_0).$$

Define $v_1 = y_1 y_2$, $v_2 = y_1/y_2$. Then $k(v_0, y_1, y_2)^{\langle \tau \rangle} = k(v_i : 0 \leq i \leq 3)$ and

$$\begin{aligned} \sigma : v_1 &\mapsto 1/v_1, \quad v_2 \mapsto -1/v_2, \quad v_0 \mapsto -v_1^2 v_0, \\ \lambda : v_1 &\mapsto v_1, \quad v_2 \mapsto 1/v_2, \quad v_0 \mapsto 1/(v_1^2 v_0). \end{aligned}$$

Define $u_1 = v_1 v_0$, $u_2 = v_2$, $u_3 = (1 - v_1)/(1 + v_1)$. Then $k(v_i : 0 \leq i \leq 2) = k(u_i : 1 \leq i \leq 3)$ and

$$\begin{aligned} \sigma : u_1 &\mapsto -u_1, \quad u_2 \mapsto -1/u_2, \quad u_3 \mapsto -u_3, \\ \lambda : u_1 &\mapsto 1/u_1, \quad u_2 \mapsto 1/u_2, \quad u_3 \mapsto u_3. \end{aligned}$$

By Theorem 2.2 $k(u_i : 1 \leq i \leq 3)^{\langle \sigma, \tau \rangle} = k(u_1, u_2)^{\langle \sigma, \tau \rangle}(u_4)$ for some u_4 with $\sigma(u_4) = \tau(u_4) = u_4$. By Theorem 2.7 $k(u_1, u_2)^{\langle \sigma, \tau \rangle}$ is rational over k . Hence $k(u_i : 1 \leq i \leq 3)^{\langle \sigma, \tau \rangle}$ is rational over k . \square

§4. The proof of Theorem 1.6

Theorem 4.1 *Let G be a group of order $4n$ where n is any positive integer. Assume that G contains a cyclic subgroup of order n . If k is a field satisfying that $\text{char } k \neq 2$, $\zeta_n \in k$ and $\sqrt{-1} \in k$, then $k(G)$ is rational over k .*

Proof. Choose $\sigma \in G$ such that the order of σ is n and write $H = \langle \sigma \rangle$. Let $G = H \cup \tau_1 H \cup \tau_2 H \cup \tau_3 H$ be a coset decomposition of G with respect to H . The idea of proving the rationality of $k(G)$ over k is very similar to that in Section 3, but it is unnecessary to specify the group action of G on the faithful representation space.

Step 1. Choose a vector X with $\sigma \cdot X = \zeta X$ where $\zeta = \zeta_n$ (see, for example, the proof of Case 1 in Section 3). Find the induced representation by defining $x_0 = X$, $x_1 = \tau_1 \cdot X$, $x_2 = \tau_2 \cdot X$, $x_3 = \tau_3 \cdot X$. For any $\lambda \in G$, any x_i where $0 \leq i \leq 3$, $\lambda \cdot x_i = \zeta^c x_j$ for some $j \in \{0, 1, 2, 3\}$ and an integer c . For example, consider $\lambda \cdot x_2$. Since $\lambda \tau_2 \in G$, it follows that $\lambda \tau_2$ belongs to one of the cosets $H, \tau_1 H, \tau_2 H, \tau_3 H$. Suppose $\lambda \tau_2 \in \tau_3 H$ and $\lambda \tau_2 = \tau_3 \sigma^c$. Then $\lambda \cdot x_2 = \lambda \cdot (\tau_2 \cdot X) = (\lambda \tau_2) \cdot X = (\tau_3 \sigma^c) \cdot X = \zeta^c x_3$.

Step 2. It is not difficult to see that G acts on the field $k(x_0, x_1, x_2, x_3)$ faithfully. For, if $\lambda \in G$, λ belongs to one of the cosets $H, \tau_1 H, \tau_2 H, \tau_3 H$. Suppose $\lambda \in \tau_1 H$ and $\lambda = \tau_1 \sigma^a$ for some integer a , then $\lambda \cdot x_0 = \tau_1 \sigma^a \cdot x_0 = \tau_1 \sigma^a \cdot X = \zeta^a \tau_1 \cdot X = \zeta^a x_1$. Thus $\lambda \cdot x_0 \neq x_0$.

Similar to the situation in Case 1 of Section 3, we may apply Theorem 2.2 to show that $k(G)$ is rational over $k(x_0, x_1, x_2, x_3)^G$. It remains to show that $k(x_0, x_1, x_2, x_3)^G$ is rational over k .

Step 3. Define $y_i = x_i/x_0$ for $1 \leq i \leq 3$. By Theorem 2.3, $k(x_0, x_1, x_2, x_3)^G = k(y_1, y_2, y_3)^G(f)$ for some polynomial $f \in k(y_1, y_2, y_3)[x_0]$. It remains to show that $k(y_1, y_2, y_3)^G$ is rational over k .

Step 4. Note that the action of G on $k(y_1, y_2, y_3)$ is a monomial action with coefficient in $\langle \zeta \rangle$. Apply Theorem 2.5 (we may apply Theorem 2.6 as well because $\sqrt{-1} \in k$). We find that $k(y_1, y_2, y_3)^G$ is rational over k . \square

Proof of Theorem 1.6. Case 1. $n = 4t$ for some integer t .

Since $n = 4t$, the assumption $\zeta_n \in k$ implies $\sqrt{-1} \in k$ also. Hence we may apply Theorem 4.1.

Case 2. n is an odd integer.

The proof is almost the same as that of Theorem 4.1, except that we don't assume that $\sqrt{-1} \in k$. The proof of Theorem 4.1 till Step 3 remains valid in the present situation. But we will apply Theorem 2.6 this time. Note that the "exceptional" case will not arise because n is odd and $-1 \notin \langle \zeta_n \rangle$.

Case 3. $n = 2m$ where m is an odd integer.

The proof is also the same as in Theorem 4.1, i.e. by using Theorem 2.6 it remains to consider the situation that there is a normal subgroup H of G such that G/H is cyclic of order 4.

Step 1. Choose $\sigma \in G$ such that the order of σ is n . Since σ^2 is of odd order m , it follows that $\sigma^2 \in H$. We will show that the cyclic subgroup $\langle \sigma^2 \rangle$ is normal in G .

Note that the 2-Sylow subgroup, say C , of H is of order 2, hence is cyclic. By Burnside's p -normal complement theorem, there is a normal subgroup H_0 of H such that H_0 is of order m and $H = H_0C$ [Is, Corollary 5.14, p.160]. In particular, H is a solvable group.

For any $\lambda \in G$, we will show that $\lambda^{-1}\sigma^2\lambda \in \langle \sigma^2 \rangle$. Since $\lambda^{-1}\sigma^2\lambda \in \lambda^{-1}H\lambda = H$, it follows that all of $H_0, \langle \sigma^2 \rangle, \langle \lambda^{-1}\sigma^2\lambda \rangle$ are subgroups of order m in H . By Hall's Theorem, they are conjugate in H [Is, Theorem 3.14]. Since H_0 is normal in H , we find these three subgroups are equal to each other. Thus $\lambda^{-1}\sigma^2\lambda \in \langle \sigma^2 \rangle$.

Step 2. We will show that the 2-Sylow subgroups of G are isomorphic to C_8 or $C_4 \times C_2$.

Let P_1 be a Sylow 2-subgroup of G containing C , which is generated by an order 2 element in H . Note P_1 is of order 8 and G is a semi-direct product of $\langle \sigma^2 \rangle$ and P_1 . It follows that $G/H \simeq P_1/C$. Thus P_1 is a group of order 8 with a cyclic quotient of order 4. Hence P_1 is not isomorphic to the dihedral group of order 8 or the quaternion group of order 8. The only possibility is that P_1 is an abelian group. In particular, $P_1 \simeq C_8$ or $C_4 \times C_2$.

Step 3. We will show that $\langle \sigma \rangle$ is a normal subgroup of G and $\sigma^m \in Z(G)$, the center of G .

Choose a 2-Sylow subgroup P_2 containing σ^m . Note that G is the semi-direct product of $\langle \sigma^2 \rangle$ and P_2 . Since P_2 is abelian, σ^m commutes with every element of P_2 (and also with σ^2). Hence it belongs to $Z(G)$.

It follows that the subgroup generated by $\langle \sigma^2 \rangle$ and $\langle \sigma^m \rangle$ is normal in G . This subgroup is nothing but $\langle \sigma \rangle$.

Step 4. By Step 2, $P_2 \simeq C_8$ or $C_4 \times C_2$.

If $P_2 \simeq C_8$, choose a generator τ of P_2 . It follows that $\tau^4 = \sigma^m$. By Burnside's Theorem again, we find that $G = N \rtimes \langle \tau \rangle$ where N is a normal subgroup of order m [Is, Corollary 5.14, p.160]. Hence $G \simeq C_m \rtimes C_8$. Using Hall's Theorem again [Is, Theorem 3.14], we find that $N = \langle \sigma^2 \rangle$.

If $P_2 \simeq C_4 \times C_2$, choose generators τ, τ_1 of P_2 so that $\langle \tau \rangle \simeq C_4$ and $\langle \tau_1 \rangle \simeq C_2$. If $\sigma^m = \tau^2$, then $G/\langle \sigma \rangle \simeq C_2 \times C_2$. If $\sigma^m = \tau_1$ or $\tau_1 \tau^2$, we may change the generators τ, τ_1 and thus we may assume that $\sigma^m = \tau_1$ with $G = \langle \sigma \rangle \rtimes \langle \tau \rangle$.

Step 5. Write $\zeta = \zeta_n$.

Subcase 1. $P_2 = \langle \tau, \tau_1 \rangle$, $\sigma^m = \tau^2$, and $G/\langle \sigma \rangle \simeq C_2 \times C_2$.

As before (see the proof of Theorem 4.1 for details), choose a vector X such that $\sigma \cdot X = \zeta X$. Define $u_0 = X$, $u_1 = \tau \cdot X$, $u_2 = \tau_1 \cdot X$, $u_3 = \tau \tau_1 \cdot X$. We find that $k(G)$ is rational over $k(u_0, u_1, u_2, u_3)^G$ and $k(u_0, u_1, u_2, u_3)^G$ is rational over $k(v_1, v_2, v_3)^G$ where $v_i = u_i/u_0$ for $1 \leq i \leq 3$.

Note that G acts on $k(v_1, v_2, v_3)$ by monomial actions with coefficients in $\langle \zeta \rangle$.

Since $\langle \sigma \rangle$ is normal in G , the action of σ on u_0, u_1, u_2, u_3 becomes very simple. We find that $\sigma(u_j) = \zeta^{b_j} u_j$ for some integer b_j . Thus $\sigma(v_j) = \zeta^{a_j} v_j$ for some integer a_j . Now apply [KPr, Lemma 2.8] (also see the proof of Theorem 2.5 and Theorem 2.6). There is a normal subgroup H of G such that $k(v_1, v_2, v_3)^H = k(z_1, z_2, z_3)$ with each one of z_1, z_2, z_3 in the form $v_1^{\lambda_1} v_2^{\lambda_2} v_3^{\lambda_3}$ (where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}$), G/H acts on $k(z_1, z_2, z_3)$ by monomial k -automorphisms, and $\rho_{\underline{z}} : G/H \rightarrow GL_3(\mathbb{Z})$ is injective.

Note that $\sigma \in \text{Kernel}\{\rho_{\underline{z}} : G \rightarrow GL_3(\mathbb{Z})\}$. Thus $\sigma \in H$. Since G/H is a quotient of $G/\langle \sigma \rangle \simeq C_2 \times C_2$, G/H is not isomorphic to the cyclic group of order 4. Now apply Theorem 2.6 to $k(z_1, z_2, z_3)^{G/H}$. We find that $k(z_1, z_2, z_3)^{G/H}$ is rational.

Subcase 2. $G = \langle \sigma \rangle \rtimes \langle \tau \rangle$ where $\langle \tau \rangle \simeq C_4$.

The proof is similar to that of Subcase 1. Choose the same vector X with $\sigma \cdot X = \zeta X$. Define $u_j = \tau^j \cdot X$ for $0 \leq j \leq 3$. The question is reduced to $k(u_0, u_1, u_2, u_3)^G$, or to $k(v_1, v_2, v_3)^G$ more simply where $v_i = u_i/u_{j-1}$ for $1 \leq i \leq 3$.

Again we have $\sigma(v_j) = \zeta^{a_j} v_j$ for some integer a_j . Moreover, since τ permutes u_0, u_1, u_2, u_3 cyclically, we find that $\tau : v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow 1/(v_1 v_2 v_3)$.

As in Subcase 1, apply [KPr, Lemma 2.8] and its proof. We find that $k(v_1, v_2, v_3)^{\langle \sigma \rangle} = k(z_1, z_2, z_3)$ where each one of z_1, z_2, z_3 is in the form $v_1^{\lambda_1} v_2^{\lambda_2} v_3^{\lambda_3}$ (with $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}$). Note that τ acts on $k(z_1, z_2, z_3)$ by purely monomial k -automorphisms, because we

have found $\tau : v_1 \mapsto v_2 \mapsto v_3 \mapsto 1/(v_1 v_2 v_3)$. Thus $k(z_1, z_2, z_3)^{\langle \tau \rangle}$ is rational by [HK1]. Done.

Subcase 3. P_2 is generated by τ with $\langle \tau \rangle \simeq C_8$ and $\tau^4 = \sigma^m$.

We will show that $k(G)$ is rational over k if and only if at least one of $-1, 2, -2$ belongs to $(k^\times)^2$.

First assume that $k(G)$ is rational. The proof is almost the same as the proof of the “exceptional case” in Theorem 2.6. The desired conclusion “at least one of $-1, 2, -2$ belongs to $(k^\times)^2$ ”, which is equivalent to “ $k(\zeta_8)$ is cyclic over k ”, is valid when k is a finite field. Hence we may assume that k is an infinite field.

Since $k(G)$ is k -rational, it is retract k -rational [Sa1, Sa2, Ka7]. By Saltman’s Theorem [Sa1, Theorem 3.1; Ka7, Theorem 3.5], we find that $k(C_8)$ is also retract k -rational because $G \simeq C_m \rtimes C_8$. Since $k(C_8)$ is retract k -rational, it follows that $k(\zeta_8)$ is cyclic over k by [Sa2, Theorem 4.12; Ka7, Theorem 2.9]. Done.

Now assume that at least one of $-1, 2, -2$ belongs to $(k^\times)^2$. We will prove that $k(G)$ is rational.

Similar to Subcase 2, choose the same vector X with $\sigma \cdot X = \zeta X$. Define $u_j = \tau^j \cdot X$ for $0 \leq j \leq 3$. The question is reduced to $k(u_0, u_1, u_2, u_3)^G$, or to $k(v_1, v_2, v_3)^G$ more simply where $v_i = u_i/u_{j-1}$ for $1 \leq i \leq 3$.

Again we have $\sigma(v_j) = \zeta^{a_j} v_j$ for some integer a_j . Moreover, since $\tau^4 = \sigma^m$, we find that $\tau : v_1 \mapsto v_2 \mapsto v_3 \mapsto -1/(v_1 v_2 v_3)$.

Apply [KPr, Lemma 2.8] and its proof. We find that $k(v_1, v_2, v_3)^{\langle \sigma \rangle} = k(z_1, z_2, z_3)$ where each one of z_1, z_2, z_3 is in the form $v_1^{\lambda_1} v_2^{\lambda_2} v_3^{\lambda_3}$ (with $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}$). Note that τ acts on $k(z_1, z_2, z_3)$ by monomial k -automorphisms with coefficients in $\langle -1 \rangle$, because we have found $\tau : v_1 \mapsto v_2 \mapsto v_3 \mapsto -1/(v_1 v_2 v_3)$.

Apply Theorem 2.6 to $k(z_1, z_2, z_3)^{\langle \tau \rangle}$. We find that $k(z_1, z_2, z_3)^{\langle \tau \rangle}$ is rational except for the situation $\tau : z_1 \mapsto z_2 \mapsto z_3 \mapsto -1/(z_1 z_2 z_3)$. But this “exceptional” case has been solved in Theorem 2.6. Hence the result. \square

It is not difficult to adapt the proof of Theorem 4.1 so as to find a short proof of Theorem 1.3 for 2-groups (but not for the p -groups of odd order in [Ka6]).

The idea of the proof in Theorem 4.1 can be applied to the M -groups defined below.

Definition 4.2 Let k be any field, G be a finite subgroup of $GL_n(k)$. G is called an M -group if every $\sigma \in G$ has exactly one non-zero entry in each column. An M -group G is related to the monomial representation which is induced from a one-dimension representation of some subgroup H of G with $[G : H] \leq n$. Since we use the terminology “monomial actions”, we avoid using the terminology “monomial representations” and we just call them M -groups.

An M -group $G \subset GL_n(k)$ acts naturally on the rational function field $k(x_1, \dots, x_n)$ as follows. For any $\sigma \in G \subset GL_n(k)$, define $\sigma \cdot x_1, \dots, \sigma \cdot x_n$ by

$$(\sigma \cdot x_1, \dots, \sigma \cdot x_n) = (x_1, \dots, x_n)\sigma,$$

i.e. if $\sigma = (a_{ij})_{1 \leq i, j \leq n} \in GL_n(k) \subset M_n(k)$, $\sigma \cdot x_j = \sum_{1 \leq i \leq n} a_{ij} x_i$.

Lemma 4.3 *Let k be any field, G be a finite M -group contained in $GL_n(k)$. Let G act on the rational function field $k(x_1, \dots, x_n)$. Then there exist a root of unity $\zeta_m \in k$, $b_1, \dots, b_n \in k \setminus \{0\}$ and such that if we define $y_i = b_i x_i$ for $1 \leq i \leq n$, then, for $\sigma \in G$, $1 \leq i \leq n$, $\sigma(y_i) = c y_j$ for some $c \in \langle \zeta_m \rangle$ and some $1 \leq j \leq n$.*

Proof. Note that, for any $\sigma \in G$, $\sigma \cdot x_i = b \cdot x_j$ for some $b \in k \setminus \{0\}$, some $1 \leq j \leq n$, i.e. if we neglect the coefficient b temporarily, σ acts on $\{x_1, x_2, \dots, x_n\}$ by some permutation. To prove this lemma, without loss of generality, we may assume that G acts on $\{x_1, \dots, x_n\}$ transitively.

For $2 \leq i \leq n$, choose $\tau_i \in G$ such that $\tau_i \cdot x_1 = b_i \cdot x_i$ where $b_i \in k \setminus \{0\}$. Define $\tau_1 = 1 \in G$ and $b_1 = 1 \in k$. Thus $\tau_i \cdot x_1 = b_i \cdot x_i$ for $1 \leq i \leq n$.

Define $y_i = b_i x_i$ for $1 \leq i \leq n$.

Define $H = \{\tau \in G : \tau \cdot x_1 = c \cdot x_1 \text{ for some } c \in k \setminus \{0\}\}$, $I = \{b \in k \setminus \{0\} : b = (\tau \cdot x_1)/x_1 \text{ for some } \tau \in H\}$. I is a subgroup of the multiplicative group $k \setminus \{0\}$. Hence I is a cyclic group and $I = \langle \zeta_m \rangle$ for some $\zeta_m \in k$.

We will prove that, for any $\sigma \in G$, any $1 \leq i \leq n$, if $\sigma \cdot y_i = c \cdot y_j$, then $c \in \langle \zeta_m \rangle$.

If $\sigma \in G$ and $(\sigma \cdot x_i)/x_j \in k$, then $(\sigma \cdot y_i)/y_j \in k$. Write $\sigma \cdot y_i = c \cdot y_j$. Plugging the formulae $\tau_i \cdot x_1 = b_i \cdot x_i$, $\tau_j x_1 = y_j$ into $\sigma \cdot y_i = c \cdot y_j$, we get $\tau_j^{-1} \sigma \tau_i(x_1) = c x_1$, i.e. $\tau_j^{-1} \sigma \tau_i \in H$ and $c \in I = \langle \zeta_m \rangle$. \square

Theorem 4.4 *Let k be a field with $\text{char } k \neq 2$ and $\sqrt{-1} \in k$. Let G be an M -group contained in $GL_4(k)$. Consider the natural action of G on the rational function field $k(x_1, x_2, x_3, x_4)$. Then the fixed field $k(x_1, x_2, x_3, x_4)^G$ is rational over k .*

Proof. By Lemma 4.3, find y_1, y_2, y_3, y_4 and $\zeta_m \in k$ such that $k(x_1, x_2, x_3, x_4) = k(y_1, y_2, y_3, y_4)$ and, for any $\sigma \in G$, any $1 \leq i \leq 4$, $\sigma(y_i) = c \cdot y_j$ for some $c \in \langle \zeta_m \rangle$, some $1 \leq j \leq 4$.

Define $z_i = y_i/y_4$ for $1 \leq i \leq 3$. Then $k(x_1, x_2, x_3, x_4)^G = k(y_1, y_2, y_3, y_4)^G = k(z_1, z_2, z_3)^G(f)$ by Theorem 2.3. Note that G acts on $k(z_1, z_2, z_3)$ by monomial k -automorphisms with coefficients in $\langle \zeta_m \rangle$. Apply Theorem 2.5. We find that $k(z_1, z_2, z_3)^G$ is rational over k . \square

Remark. If $\sqrt{-1}$ doesn't belong to the base field k , it may happen that the conclusion of Theorem 4.4 may fail. For example, consider an action $\sigma : x_1 \mapsto x_2 \mapsto x_3 \mapsto x_4 \mapsto -x_1$; the fixed field $\mathbb{Q}(x_1, x_2, x_3, x_4)^{\langle \sigma \rangle}$ is not rational over \mathbb{Q} by [Ka1, Theorem 1.8].

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